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Some Results for Probability Measures on Linear Topological Vector Spaces with an Application to Strassen's Log Log Law

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We prove some results regarding tight probability measures on real Frechet spaces and countable strict inductive limits of these spaces. These results are applied to Gaussian measures and to construct a Brownian motion on such spaces. We then prove the log log law of Strassen for this Brownian motion.

INTRODUCTION

Throughout E is a real Hausdorff locally convex topological vector space. If the topology of E is metrizable and complete, then E is called a Frechet space, and it is well known that the topology on E is generated by an increasing sequence of semi-norms $\|\cdot\|_j$ ($j = 1, 2,...$) such that

$$\|\cdot\|_E = \sum_{j=1}^{\infty} 2^{-j}\|\cdot\|_j/(1 + \|\cdot\|_j) \tag{1.1}$$

gives an invariant metric on E which generates the topology of E.

The topological dual of E is denoted by E' and as usual the Borel subsets of E denote the minimal sigma-algebra containing the open sets. A Borel probability measure μ is tight if for each $\epsilon > 0$ there exists a compact subset K_ϵ such that $\mu(K_\epsilon) > 1 - \epsilon$. μ is regular if for each $\epsilon > 0$ and each Borel set A there is some compact set $K \subseteq A$ such that $\mu(A \cap K^c) < \epsilon$.

If E is a Frechet space it is not difficult to show that if μ is tight on E, then μ sits on a closed separable subspace of E. On the other hand, if E is a separable Frechet space then it is well known that all Borel probability measures on E are regular.

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In Theorem 1 and its corollary we show that every tight probability measure on a real Frechet space sits on a real separable Banach subspace of E, and if μ has certain absolute continuity properties we extend this result to countable strict inductive limits of real Frechet spaces.

A Borel probability measure is a mean zero Gaussian measure if each continuous linear functional on E has a Gaussian distribution with mean zero.

The basic structure of mean zero Gaussian measures on E is given in [2, Theorem 4], and in Theorem 2 and Theorem 3 we obtain slightly more detailed information. This information is then used to define a Brownian motion on E and to prove the law of the iterated logarithm of Strassen for this Brownian motion.

We also provide an example which developed through discussions with R. M. Dudley and the basic idea behind it is due to him. This example shows that Theorem 1 cannot be extended to an arbitrary complete locally convex Hausdorff topological vector space even if we restrict μ to be a mean zero Gaussian measure.

2. Some Results Regarding Tight Probability Measures

Let E be a vector space over the reals and assume $E = \bigcup_{n=1}^{\infty} E_n$ where $\{E_n : n \geqslant 1\}$ is an increasing sequence of linear subspaces of E. Further, assume that each E_n is a Frechet space such that the topology induced by E_{n+1} on E_n is identical to the topology initially given on E_n. Given the sequence of subspaces $\{E_n\}$ we define a locally convex Hausdorff topology on E called the strict inductive limit topology by saying a convex subset V of E is a neighborhood of zero if $V \cap E_n$ is a neighborhood of zero in E_n for $n = 1, 2,...$. When we provide E with this topology we call E a countable strict inductive limit of Frechet spaces $\{E_n\}$.

Theorem 1. *Suppose E is a strict inducitve limit of Frechet spaces $\{E_n\}$ and μ is a tight probability measure on E such that the distribution of each linear functional is degenerate at a point or absolutely continuous with respect to Lebesgue measure. Then there exists an N such that $\mu(E_N) = 1$, and a Banach space $B \subseteq E_N \subseteq E$ with norm $\|\cdot\|_0$ such that*

(2.1) *B is a real separable Banach space in $\|\cdot\|_0$.*

(2.2) *The identity map of B into E_N is continuous, and maps Borel* *subsets of B to Borel subsets of E_N. In fact, the Borel subsets of B are precisely the Borel subsets of E_N intersected with B.*

(2.3) $\mu(B) = 1$ *and hence* μ *determines a unique probability measure on the Borel subsets of B.*

Proof. Choose $\epsilon > 0$ and assume K is a compact subset of E such that $\mu(K) > 1 - \epsilon$. Then $K \subseteq E_N$ for some N [5, p. 164] and since E_N is closed in E we have $\mu(E_N) > 1 - \epsilon$. Now under the conditions on μ we actually have $\mu(E_N) = 1$. That is, if $\mu(E_N) = 1 - \delta < 1$, then there exists a compact set $C \subseteq E_n (n > N)$ such that $\mu(C) > 1 - \delta$. Since the topology induced on E_n makes E_n a Frechet space and E_N is a closed subset of E_n we have a $\gamma > 0$ such that if d is the metric on E_n and $\Lambda = \{y \in E_n : d(y, E_N) \geqslant \gamma\}$ then $\mu(C \cap \Lambda) > 0$. Now by the Hahn–Banach theorem for each point $y \in C \cap \Lambda$ there exist an $f_y \in E'$ such that $f_y(E_N) = 0$ and $f_y(y) \neq 0$. Thus, $U_y = \{x: |f_y(x)| > 0\}, y \in C \cap \Lambda$, is an open cover of the compact set $C \cap \Lambda$, and, hence, there is a y such that $| f_y |$ is positive with positive μ-probability. Hence, the distribution of f_y contradicts our hypothesis since it has positive mass at zero but it is not degenerate. Thus, $\mu(E_N) = 1$.

Since E_N is a Frechet space there are countably many semi-norms $\| \cdot \|_1 \leqslant \| \cdot \|_2 \leqslant \cdots$ such that (1.1) holds and $\| \cdot \|_{E_N}$ generates an invariant metric giving the topology on E_N. Now μ tight implies there is an increasing sequence of compact sets $K_1 \subseteq K_2 \subseteq \cdots$ in E_N such that $\mu(E_N - K_n) < 1/n$. Letting $K_0 = \bigcup_{n=1}^{\infty} K_n$ we have $\mu(K_0) = 1$. Since each K_n is compact and metrizable there is a countable dense subset $\{r_{n,j} : j \geqslant 1\}$ for K_n. Let $R = \bigcup_{n=1}^{\infty} \{r_{n,j} : j \geqslant 1\}$. Then R is a countable dense subset of K_0 with respect to the topology induced on K_0, and all finite linear combinations of elements of R with rational coefficients form a countable dense subset of the subspace F consisting of all finite linear combinations of elements of K_0. The closure $\bar{F}$ of F in E_N is then a separable Frechet space and $\mu(\bar{F}) = 1$.

Now we produce the Banach space B and the norm $\| \cdot \|_0$. Let $M_j = 1 + \sup_{x \in K_j} \| x \|_j$. Then $M_j < \infty$ since $\| \cdot \|_j$ is continuous on E and K_j is compact. Let $a_j = b_j / M_j$, where $b_j > 0$ and $\sum_{j=1}^{\infty} b_j < \infty$. Define

$$\| x \|_0 = \sum_{j=1}^{\infty} a_j \| x \|_j . \tag{2.4}$$

Then $x \in K_0 \Rightarrow x \in K_n$ for all $n > n_0 \Rightarrow \| x \|_n \leqslant M_n$ for all

$$n > n_0 \Rightarrow \| x \|_0 \leqslant \sum_{j=1}^{n_0} a_j \| x \|_j + \sum_{j=n_0+1}^{\infty} b_j < \infty.$$

Thus, $\|\cdot\|_0$ is finite on K_0 and, hence, on F. Since the semi-norms $\{\|\cdot\|_j : j \geqslant 1\}$ separate points of E_N we, thus, have $\|\cdot\|_0$ a norm on F. Further, if $\{x_n\} \subseteq F$ and $\{x_n\}$ is Cauchy in $\|\cdot\|_0$ then $\{x_n\}$ is Cauchy in E_N. Hence, if B is the completion of F in the norm $\|\cdot\|_0$ we can identify B with a subset of E_N and the identity map of B into E_N is continuous. That Borel subsets of B are mapped via the identity to Borel subsets of E_N follows immediately by [1, p. 123]. Hence, the Borel subsets of B are contained in the sigma-algebra consisting of the Borel subsets of E_N intersected with B. However, the class of all subsets of E_N whose intersection with B is a Borel subset of B forms a sigma-algebra of E_N, and it contains the E_N-open sets since the topology induced on B is weaker than the $\|\cdot\|_0$ topology, i.e., the identity map is continuous. Hence, (2.2) holds.

Now B is separable in $\|\cdot\|_{E_N}$ and to see B is separable in $\|\cdot\|_0$ it suffices to show R is $\|\cdot\|_0$ dense in K_0. Fix $\epsilon > 0$. Then $x \in K_0 \Rightarrow x \in K_n$ for all

$$n \geqslant n_0 \Rightarrow \| x - p_{n,k} \|_0 \leqslant \sum_{j=1}^{N} a_j \| x - p_{n,k} \|_j + \sum_{j=N+1}^{\infty} 2b_j$$

for all $N \geqslant n \geqslant n_0$ since $\{p_{n,k}\} \subseteq K_n \subseteq K_r$ implies $\| x - p_{n,k} \|_r \leqslant 2 M_r$ for all $r \geqslant n \Rightarrow \| x - p_{n,k} \|_0 \leqslant \sum_{j=1}^{N} a_j \| x - p_{n,k} \|_j + \epsilon/2$ for some $N \geqslant n$ and any $n \geqslant n_0 \Rightarrow \| x - p_{n,k} \|_0 < \epsilon$ if we choose $p_{n,k}$ such that $a_j \| x - p_{n,k} \|_j < \epsilon/2N$ and this is possible. Hence, (2.1) holds.

Finally, (2.3) is obvious since $B \supseteq \bigcup_{n=1}^{\infty} K_n$ and B is a Borel subset of E_N.

COROLLARY 1. *Let μ be a tight Borel probability measure on a real Frechet space E. Then there exists a subspace B of E and a norm $\|\cdot\|_0$ defined on B such that* (2.1), (2.2) *and* (2.3) *hold for B with E_N replaced by E.*

Proof. Since E is a Frechet space this result follows from Theorem 1 and its proof since the absolute continuity requirements on μ in Theorem 1 were only used to prove $\mu(E_N) = 1$ for some N.

It is easy to see that the conditions in Theorem 1 regarding the distribution of every linear functional cannot be eliminated. For example, let E denote the strict inductive limit of the sequence of spaces $\{E_n\}$ where each E_n is real n-dimensional Euclidean space with canonical basis $\{e_1, \ldots, e_n\}$, and let $\mu\{e_j\} = 1/2^j$ for $j = 1, 2, \ldots$. Then μ is a tight probability measure on E, and if B is any Banach space within E such that (2.1), (2.2), and (2.3) hold then (2.3) implies

$B \supseteq \{e_j : j \geqslant 1\}$ and, hence, $B = E$ as a set of points. Thus, $B = \bigcup_{n=1}^{\infty} E_n$ and this is a contradiction since each E_n is a closed nowhere dense subset of B.

3. A Counterexample for Theorem 1

This example shows that Theorem 1 can not be extended to an arbitrary complete locally convex Hausdorff topological vector space even if we restrict μ to be a mean zero Guassian measure.

Let $\mathbb{R}^\infty$ denote the space of all real sequences with the product topology, and let $H = l_2$ with the usual Hilbert space structure. Let $\mathscr{E}$ denote the set of all decreasing sequences of positive numbers $\{\epsilon_n\}$ such that $1 \geqslant \epsilon_n \downarrow 0$. For each $\epsilon = \{\epsilon_n\}$ in $\mathscr{E}$ let $\|\cdot\|_\epsilon$ denote the norm defined on H by

$$\| x \|_\epsilon = \sup_n \frac{|\epsilon_n x_n|}{(\log(n+1))^{1/2}}$$

where $x = \{x_n\}$. Let M denote all sequences $\{x_n\}$ in $\mathbb{R}^\infty$ such that

$$\sup_n \frac{|x_n|}{(\log(n+1))^{1/2}} < \infty.$$

Lemma. *If S is the locally convex topology defined on M by the norms $\|\cdot\|_\epsilon$ for ϵ in $\mathscr{E}$ then*

(1) *M is a complete Hausdorff locally convex topological vector space in S, and*

(2)

$$D = \left\{\{x_n\}: \sup_n \frac{|x_n|}{(\log(n+1))^{1/2}} \leqslant 1\right\}$$

is a compact set in M with the topology S.

Proof. Let $\{p_\alpha\}$ denote a Cauchy net in M and suppose π_n is the projection of a sequence $x = \{x_i\}$ onto its nth coordinate. Now $\{p_\alpha\}$ a Cauchy net in M implies that for every norm $\|\cdot\|_\epsilon$, ϵ in $\mathscr{E}$,

$$\lim_{\alpha,\beta} \| p_\alpha - p_\beta \|_\epsilon = 0,$$

and, hence,

$$\lim_{\alpha,\beta} | \pi_n(p_\alpha) - \pi_n(p_\beta) | \leqslant \lim_{\alpha,\beta} \frac{(\log(n+1))^{1/2}}{\epsilon_n} \| p_\alpha - p_\beta \|_\epsilon = 0.$$

Thus, $\pi_n(p_\alpha)$ is a convergent net of real numbers with limit, say y_n, and for each ϵ in $\mathscr{E}$

$$\sup_n \frac{|y_n\epsilon_n|}{(\log(n+1))^{1/2}} = \sup_n \lim_\alpha \frac{|\pi_n(p_\alpha)\epsilon_n|}{(\log(n+1))^{1/2}}$$
$$\leqslant \lim_\alpha \| p_\alpha \|_\epsilon < \infty, \tag{3.1}$$

since $\{p_\alpha\}$ is Cauchy. Furthermore, $y = \{y_n\}$ is in M. That is, if $\sup_n |y_n|/(\log(n+1))^{1/2} = \infty$ then there exists a subsequence $\{n_k\}$ such that $1 \leqslant |y_{n_k}|/(\log(n_k+1))^{1/2} \uparrow \infty$; and, hence, if $\epsilon_n = ((\log(n_{k+1}))^{1/2}/|y_{n_k}|)^{1/2}$ for $n_{k-1} < n \leqslant n_k$ with $n_0 = 0$ then $\epsilon = \{\epsilon_n\}$ is in $\mathscr{E}$ and

$$\| y \|_\epsilon = \sup_n \frac{|y_n\epsilon_n|}{(\log(n+1))^{1/2}} \geqslant \sup_k \left(\frac{|y_{n_k}|}{(\log(n_k+1))^{1/2}}\right)^{1/2} = +\infty.$$

This contradicts (3.1) so $y = \{y_n\} \in M$. Now

$$\lim_\alpha \| p_\alpha - y \|_\epsilon \leqslant \lim_{\alpha_0} \sup_{\alpha \geqslant \alpha_0} \sup_n \frac{|\pi_n(p_\alpha) - \pi_n(y)|\epsilon_n}{(\log(n+1))^{1/2}}$$
$$= \lim_{\alpha_0} \sup_{\alpha \geqslant \alpha_0} \sup_n \lim_\beta \frac{|\pi_n(p_\alpha) - \pi_n(p_\beta)|\epsilon_n}{(\log(n+1))^{1/2}}$$
$$\leqslant \lim_{\alpha_0} \sup_{\alpha,\beta \geqslant \alpha_0} \sup_n \frac{|\pi_n(p_\alpha) - \pi_n(p_\beta)|\epsilon_n}{(\log(n+1))^{1/2}}$$
$$= \lim_{\alpha_0} \sup_{\alpha,\beta \geqslant \alpha_0} \| p_\alpha - p_\beta \|_\epsilon = 0,$$

and, hence, $\{p_\alpha\}$ converges to y for each norm $\|\cdot\|_\epsilon$. Thus, $\{p_\alpha\}$ converges to y in M and M is complete.

To show D is a compact we first show D is closed (and, hence, complete). Suppose $\{p_\alpha\}$ is a net in D and $p_\alpha \to y$ in M. If

$$\sup \frac{|y_n|}{(\log(n+1))^{1/2}} > 1$$

then there exists an N such that $|y_N|/(\log(N+1))^{1/2} > 1$ and since $\lim_N \pi_N p_\alpha = y_N$ we have

$$\lim_\alpha \frac{|\pi_N(p_\alpha)|}{(\log(N+1))^{1/2}} > 1.$$

This is a contradiction since $\sup_n |\pi_N(p_\alpha)|/(\log(n+1))^{1/2} \leqslant 1$ for each α, and, hence, $y = \{y_n\} \in D$.

Finally we show D is compact. By applying [5, Theorems 7.6 and 7.7, p. 61] we need only show that each sequence in D has a cluster point in D. Let $\{p_n\} \subseteq D$. Then there exists a subsequence $\{p_{n_k}\}$ such that $\lim_k \pi_N p_{n_k} = y_N$ for $N = 1, 2, \ldots$. Further,

$$\sup_N \frac{|y_N|}{(\log(N+1))^{1/2}} = \sup_N \lim_k \frac{|\pi_N(p_{n_k})|}{(\log(N+1))^{1/2}} \leqslant \sup_{N,k} \frac{|\pi_N(p_{n_k})|}{(\log(N+1))^{1/2}} \leqslant 1$$

so $y = \{y_N\} \in D$ and for each ϵ in $\mathscr{E}$ we have

$$\begin{aligned} \lim_k \| p_{n_k} - y \|_\epsilon &= \lim_k \sup_N \frac{|\pi_N(p_{n_k}) - \pi_N(y)| \epsilon_N}{(\log(N+1))^{1/2}} \\ &\leqslant \lim_k \sup_{N<N_0} \frac{|\pi_N(p_{n_k}) - \pi_N(y)| \epsilon_N}{(\log(N+1))^{1/2}} + \sup_{N \geqslant N_0} \epsilon_N \\ &\leqslant \epsilon_{N_0} . \end{aligned}$$

Since $\epsilon_{N_0} \downarrow 0$ we have p_{n_k} converging to y for each norm $\| \cdot \|_\epsilon$, and hence, p_{n_k} converges to y in M. Thus, D is compact and the lemma is proved.

Let v denote the mean zero Gaussian measure on $\mathbb{R}^\infty$ such that $\{\pi_N(x): N \geqslant 1\}$ is a sequence of independent Gaussian random variables with mean zero and variance one. Then M and D are Borel subsets of $\mathbb{R}^\infty$ and $v(M) = 1$. Since $M = \bigcup_{n=1}^\infty nD$ we have $\sup_n v(nD) = 1$ and nD is compact in M. Since $H = l_2$ maps continously into M under the identity we have the topological dual of M, call it M', mapped into $H' = H$. Hence, v induces a mean zero Gaussian cylinder set measure m on the cylinder sets of M given by M'. By [2, Theorem 1] m extends to a unique regular mean zero Gaussian measure μ on M.

If B is a separable Banach space such that $B \subseteq M$, B maps continuously into M under the identity, and $\mu(B) = 1$, then μ is defined on the Borel subsets of B [1, p. 123] and, hence, is a regular mean zero Gaussian measure on the Borel subset of B. Furthermore, the Hilbert space generating μ on B is $H = l_2$. Now $\| \cdot \|_B$ continuous on H in the S topology implies there exists an ϵ in $\mathscr{E}$ such that

$$\| x \|_B \leqslant C \| x \|_\epsilon \qquad (x \in H),$$

and, hence, the completion of H under $\| \cdot \|_B$ (which is a subset of B) contains all sequences of the form $\{x_n\}$, where

$$\sup_n \frac{|x_n \epsilon_n|}{(\log(n+1))^{1/2}} < \infty.$$

This is a contradiction, since $B \subseteq M$. Thus, μ is a regular mean zero Gaussian measure on a complete locally convex Hausdorff topological vector space and there is no separable Banach space B as in Theorem 1 such that $\mu(B) = 1$.

4. Gaussian Measures

The support of a Borel probability measre μ on a Hausdorff topological space E is defined as the set of all points p in E such that every open set containing p has positive μ measure. Obviously the support of μ is a closed set.

Theorem 2. *Let E be a complete locally convex Hausdorff topological vector space and suppose μ is a mean zero regular Gaussian measure on the Borel subsets of E. Then*

(a) *there is a separable Hilbert space H such that $H \subseteq E$ and H maps continuously into E under the identity map, and*

(b) *if M is the closure of H in E, then $\mu(M) = 1$, and the support of μ is M (thus the support of μ is a separable topological vector space).*

Proof. Let $\mathscr{H}$ be the closure in $L^2(\mu)$ of the random variables $f(\cdot), f \in E'$ and let $\psi : E' \to \mathscr{H}$ be defined by $\psi(f) = f(\cdot)$. In view of the tightness of μ it then follows as in Theorem 4 of [2] that the transpose of ψ, call it θ, is a one-to-one continuous linear map from $\mathscr{H}$ into E. We define $H = \theta(\mathscr{H})$ and define the inner product on H by setting $(\theta(x_1), \theta(x_2))_H = (x_1, x_2)_{\mathscr{H}}$.

Let $\{f_\alpha : \alpha \in A\}$ denote a complete orthonormal basis for $\mathscr{H}$ such that each $f_\alpha \in E'$. Then $e_\alpha = \theta(f_\alpha)$ $(\alpha \in A)$ is a complete orthonormal basis for H, and to verify (a) it suffices to show A is countable since we already know θ is continuous from $\mathscr{H}$ into E.

Now f_α $(\alpha \in A)$ orthonormal in $\mathscr{H}$ implies the f_α's are independent Gaussian random variables with mean zero and variance one. Let $W = \prod_{\alpha \in A} \mathbb{R}_\alpha$, where $\mathbb{R}_\alpha = (-\infty, \infty)$ and topologize W with the product topology. Let $\Lambda: E \to W$ be defined by

$$\Lambda(x) = (f_\alpha(x): \alpha \in A).$$

Then Λ is a continuous map from E into W and if $\mu^\Lambda(C) = \mu(\Lambda^{-1}(C))$ where C is a Borel subset of W then μ^Λ is a mean zero Gaussian measure on W. Now μ tight on E and Λ continuous implies μ^Λ is tight on W. However, this is possible only if A is countable. That is, since the $\{f_\alpha : \alpha \in A\}$ are independent and Gaussian it follows that μ^A is the product measure on W formed by the product of Gaussian measures on $(-\infty, \infty)$ with mean zero and variance one. Now K compact in W implies $K \subseteq \prod_{\alpha \in A} [u_\alpha, v_\alpha]$ where $-\infty < u_\alpha \leqslant v_\alpha < \infty$ for each $\alpha \in A$, and hence $\mu^A(K) = 0$ unless A is countable. Hence, (a) holds.

Let M be as in (b) and suppose F is a continuous linear functional which vanishes on M. Then F restricted to H is identically zero, and since $\psi(F) = F|_H$ (F restricted to H) we see the distribution of F is degeneate at zero. Thus, by the Hahn–Banach theorem each point $p \in E - M$ has an open neighborhood with zero μ-measure. If $\mu(M) < 1$, then μ being regular implies that there is a compact set $K \subseteq E \cap M^c$ such that $\mu(K) > 0$. Now this is impossible since we then have K covered by a finite union of open sets with zero measure. Hence, $\mu(M) = 1$ and the support of μ, call it S, is in M. Since S is closed by definition if $S \neq M$, then there exists an open set U of E such that $\mu(U) = 0$, $U \cap S = \phi$, and an $h \in H \subseteq M$ such that $h \in U$. Since H is mapped continuously into E under the identify map and H is separable it follows that for any countable dense $\{h_j\}$ of H we have $M \subseteq \bigcup_{j=1}^{\infty} (U + h_j)$. That is, since $U - h$ is an open set containing zero there is a balanced open set V containing zero such $V + V \subseteq U - h$, and, hence, $x \in M \Rightarrow$ there exists an $h_0 \in H$ such that $x - h_0 \in V \Rightarrow x \in V + h_0 \subseteq V + h_0 - (h + h_j) + (h_j + h) \subseteq V + V + (h_j + h) \subseteq U + h_j$ if h_j is taken so that $h_0 - h + h_j \in V$. Now it is well known that μ translated by any $h \in H$ (see [9, p. 22]) is equivalent to μ, and, hence, $\mu(M) \leqslant \sum_{j=1}^{\infty} \mu(U + h_j) = 0$. Contradiction, so we have $S = M$ and the theorem is proved.

THEOREM 3. *Let μ be a tight mean zero Gaussian measure on E where E is a strict inductive limit of Frechet spaces $\{E_n\}$. Then there exists a Hilbert space $H \subseteq E$ such that*

(a) *H is a separable Hilbert space with norm $\|\cdot\|_H$ and $H \subseteq E_N$ for some integer N.*

(b) *The closure of H in E_N (and, hence, in E) is a separable subspace of E_N of μ-measure one.*

(c) *The identity map of H into E_N is continuous. If Γ is the map which restricts an element in E' to H, then Γ is linear and $\Gamma(E')$ is a dense linear subspace of H'. Hence, if we identify H' and H (as we do) then $\Gamma(E')$ can be viewed as a dense subset of $H \subseteq E$. Further, Γ satisfies $\|\Gamma(f)\|_H^2 = \int_E |f(x)|^2 \, d\mu(x)$ for $f \in E'$.*

(d) *The set H is uniquely determined since it is precisely the set* *of translates of μ which yield a measure equivalent to μ. Further, the norm on H is uniquely determined by the formula for $\| \Gamma(f)\|_H$ given in* (c).

(e) *If $\{e_k\}$ is a complete orthonormal sequence for H which is a subset of E' then*

$$\lim_N \left\| x - \sum_{k=1}^{N} (x, e_k)e_k \right\| = 0 \tag{4.1}$$

with μ-measure one for every continuous semi-norm on E_N (on E) and in particular (4.1) *holds for the invariant metric $\| \cdot \|_{E_N}$ determined as in* (1.1). *Here $(x, e_k) = e_k(x)$.*

(f) *If m_t $(t > 0)$ is the canonical Gaussian cylinder set measure on H restricted to the cylinder sets of E induced by E' then each m_t extends to a unique regular mean zero Gaussian measure μ_t on E such that $\mu_t(E_N) = 1$. Further, $\mu = \mu_1$, the convolution $\mu_s * \mu_t$ equals μ_{s+t} for all $s, t > 0$, and $\mu_t(A) = \mu(A/t^{1/2})$.*

Proof. By Theorem 1 there is an integer N such that $\mu(E_N) = 1$ and a Banach space $B \subseteq E_N \subseteq E$ normed by $\| \cdot \|_0$ such that (2.1), (2.2), and (2.3) hold for B and E_N. Then μ is a mean zero tight Gaussian measure on E_N and, in fact, on B. That is, by [4, Lemma 3.1] or by extending the arguments used in [6, Theorem 3.1] we have that if T is a linear functional on E_N which is measurable with respect to the μ-completion of the Borel subsets of E_N, then T has a mean zero Gaussian distribution. Now if f is a continuous linear functional on B, then f can be extended to be linear on all of E_N, and since $\mu(B) = 1$ and f continuous on B we have that f (extended) is measurable with respect to the μ-completion of the Borel subsets of E_N and hence f has a mean zero Gaussian distribution. Thus, μ is a mean zero Gaussian measure on B.

Now (a), (b), (c) follow directly from Theorem 2 applied to μ on E_N.

That the norm on H is uniquely determined as indicated is obvious since $\Gamma(E')$ is dense in $H' = H$. Now the set H (as a subset of B) is unique since it represents the set of translates of μ in B which yield a measure equivalent to μ (see, for example, [8, p. 357]. Hence, we need only show H is independent of B since given any vector $c \in F_N$ we can get a Banach space as in Theorem 1 which contains c (simply take the balanced convex hull of c and the compact set K_n to obtain a compact set K_n' and proceed as in Theorem 1). Thus, assume B_1 and B_2 are Banach subspaces of E_N obtained from μ as in Theorem 1 and H_1 and H_2 are the related Hilbert spaces obtained in (a). If

$b \in H_2$ then μ translated by b is equivalent to μ, and $b \in B_1$ or else $(b + B_1) \cap B_1 = \phi$ which implies

$$1 = \mu(E_N) \geqslant \mu(b + (B_1 \cap B_2)) + \mu(B_1 \cap B_2) = 2.$$

Thus, $H_2 \subseteq B_1$ so $H_2 = H_1$ since H_1 is precisely the set of translates of μ in B_1 yielding a measure equivalent to μ. Hence, (d) holds.

Let $\{e_k\}$ and $\| \cdot \|$ be as in (e). Then $\| \cdot \|$ is a continuous semi-norm on E_N or on E and the identity of B into E_N (or E) being continuous implies there is a constant C such that $\| \cdot \| \leqslant C \| \cdot \|_0$ on B. Now [11, Theorem 3.1] implies (4.1) for $\| \cdot \|_0$, and, hence, (4.1) with μ-measure one for $\| \cdot \|$. Further, since only countably many continuous semi-norms determine the invariant metric $\| \cdot \|_{E_N}$ we have (4.1) holding for $\| \cdot \|_{E_N}$,

Now we turn to (f). First we observe that $\| \cdot \|_0$ is a measurable norm on H in the sense used in [3]. This follows from a number of known results. For example, by [7, Lemma 2] the identity map of H into B is continuous, and since m_1 extends to $\mu_1 = \mu$ on B we have by [2, Theorems 2 and 3] that $\| \cdot \|_0$ is a measurable norm on H. However, one can also prove this by showing that $\| \cdot \|_0$ is the limit of an increasing sequence of tame semi-norms $\| \cdot \|_n$ on H such that

$$\lim_n m_1(x \in H : \| x \|_n \leqslant \epsilon) > 0.$$

Then by [3] we have $\| \cdot \|_0$ a measurable norm on H.

Since $\| \cdot \|_0$ is a measurable norm on H we then have by [3] that each m_t restricted to cylinder sets of B can be extended to a unique regular mean zero Gaussian measure μ_t on B such that $\mu_1 = \mu$, $\mu_s * \mu_t = \mu_{s+t}$ for all $s, t > 0$, and $\mu_t(A) = \mu(A/t^{1/2})$. Thus, by [2, Theorem 1] each m_t restricted to the cylinder sets of E induced by E' has a unique regular extension to a mean zero Gaussian measure on E, call it λ_t. Now B satisfies (2.2), and, hence, μ_t is a regular mean zero Gaussian measure on E which extends m_t. Thus, by the uniqueness of λ_t we have $\lambda_t = \mu_t$ so (f) holds.

If μ is a tight mean zero Gaussian measure on a strict inductive limit E of Frechet spaces $\{E_n\}$ and H is the unique Hilbert space given in Theorem 2, then we say μ *is generated by* H. That is, μ is the unique regular extension to E of the cylinder set measure m_1 given in (f) of Theorem 2.

5. Brownian Motion in E and Some of Its Properties

Let Ω_E denote the space of continuous functions w from $[0, \infty)$ into the strict inductive limit E of the sequence of Frechet spaces $\{E_n\}$ such that $w(0) = 0$, and let $\mathscr{F}$ be the sigma-algebra of Ω_E generated by the functions $w \to w(t)$. Let μ be a tight mean zero Gaussian measure on E generated by H, and suppose $\{\mu_t : t \geqslant 0\}$ is the unique family of regular Gaussian measures on E given in (f) of Theorem 2 when $t > 0$ and μ_0 is the unit mass at zero. Here we prove there is a unique probability measure P on $\mathscr{F}$ such that if $0 = t_0 < t_1 < \cdots < t_n$ then $w(t_j) - w(t_{j-1})$ $(j = 1,..., n)$ are independent and $w(t_j) - w(t_{j-1})$ has distribution $\mu_{t_j - t_{j-1}}$ on E. The stochastic process $\{W_t : t \geqslant 0\}$ defined on $(\Omega_E, \mathscr{F}, P)$ by $W_t(w) = w(t)$ has stationary independent mean zero Gaussian increments and we call it the *Brownian motion in E generated by μ*. In the case E is a real separable Banach space the existence of a Brownian motion was discussed in [3], and the next theorem proves its existence for an arbitrary strict inductive limit or Frechet spaces $E = \{E_n\}$.

Let F be a topological vector space, and let C_F denote the continuous functions on $[0, 1]$ into F which vanish at zero.

If F is a locally convex Hausdorff topological vector space whose topology is generated by the semi-norms $\{\| \cdot \|_\alpha : \alpha \in A\}$, then we make C_F into a locally convex Hausdorff space in the topology generated by the semi-norms $\{\| f \|_{\alpha,\infty} : \alpha \in A\}$, where $\| f \|_{\alpha,\infty} = \sup_{0 \leqslant t \leqslant 1} \| f(t) \|_\alpha$. It is easy to see that the topology on C_F is independent of the family of semi-norms used to generate the given topology on F. Further, if F is a Frechet space whose topology is generated by the increasing sequence of semi-norms $\| \cdot \|_j$ then we make C_F into a Frechet space in the locally convex topology generated by the sequence of semi-norms $\| f \|_{j,\infty} = \sup_{0 \leqslant t \leqslant 1} \| f(t) \|_j$, and if N is a closed subspace of F, then C_N is a closed subspace of C_F. If E is a strict inductive limit of the Frechet spaces $\{E_N\}$ then we have $C_E = \bigcup_{n=1}^{\infty} C_{E_n}$ since we know each compact set in E (in particular, every continuous image of $[0, 1]$ into E) must be a compact subset of some E_n. Further, the topology induced on C_{E_n} by $C_{E_{n+1}}$ is that originally given for C_{E_n} since E_{n+1} induces on E_n the given topology on E_n. Hence, we make C_E into a strict inductive limit of the Frechet spaces $\{C_{E_n}\}$.

If the Brownian motion on E generated by μ exists then the measure P induced on C_E by Brownian motion is called the *Wiener process or Wiener measure generated by μ*.

Theorem 4. *If μ is a tight mean zero Gaussian measure on a strict* *inductive limit of Frechet spaces $E = \{E_n\}$ generated by the Hilbert space H then*

(a) *the Brownian motion on E generated by μ exists and if M is the closure of H in E then M is a closed separable subspace of some E_N and our Brownian motion is, with probability one, in M.*

(b) *C_M is a separable Frechet space and the minimal sigma-algebra of C_M making the mappings $f \to f(t)$ measurable consists of the Borel subsets of C_M .*

(c) *the Brownian motion on E generated by μ induces a regular mean zero Gaussian measure P on C_E such that the subspace C_M satisfies $P(C_M) = 1$.*

(d) *for every complete orthonormal sequence $\{e_n\}$ in H which is a subset of E' and any continuous semi-norm on M we have*

$$\lim_N \sup_{0<t<1} \left\| w(t) - \sum_{k=1}^{N} (w(t), e_k)e_k \right\| = 0, \tag{5.1}$$

with P-probability one. Further, we have the above partial sums converging to $w(t)$ with P-probability one in C_E .

(e) *Let $\mathscr{H}$ denote the Hilbert space in the strict inductive limit space C_E which generates P on C_E . Then for any complete orthonormal set $\{e_j : j \geqslant 1\}$ in H which is a subset of E' we have $f \in \mathscr{H}$ iff*

(i) $f(t) = \sum_j \int_0^t (d/ds)(f(s), e_j)\, ds\, e_j$ *for each t in* $[0, 1]$ *where the series converges in H, and*

(ii) $\| f \|_{\mathscr{H}}^2 = \sum_j \int_0^1 [d(/ds)(f(s), e_j)]^2\, ds < \infty$.

(f) *If $\mathscr{K}$ is the unit ball of $\mathscr{H}$ then $\mathscr{K}$ is a compact subset of C_E .*

Proof. Given μ, let B, $\|\cdot\|_0$ denote the separable Banach space obtained as in the proof of Theorem 1 and assume $B \subseteq E_N \subseteq E$. Then by (f) of Theorem 3 we have $\mu_t(B) = 1$ for all $t \geqslant 0$.

Now let Ω_B denote the continuous functions on $[0, \infty)$ into B which vanish at zero. Then $\Omega_B \subseteq E_{E_N} \subseteq \Omega_E$ and we also have $\Omega_B \in \mathscr{F}$, where $\mathscr{F}$ is the minimal sigma-algebra of Ω_E making the maps $w \to w(t)$ measurable. That is, let

$$F_n = \{w \in \Omega_E \colon w \text{ is continuous on } [0, n] \text{ into } B\}$$

for $n = 1, 2, \ldots$. Then $\Omega_B = \bigcap_{n=1}^{\infty} F_n$, and, hence, $\Omega_B \in \mathscr{F}$ if each

$F_n \in \mathscr{F}$. Let $\{t_j\}$ be a countable dense subset of $[0, n]$. Then, since B is complete and $[0, n]$ is compact.

$$F_n = \{w \in \Omega_E: w(t_j) \in B \ (j = 1, 2,\ldots) \text{ and } \lim_{N\to\infty} \sup_{|t_k - t_l| < 1/N} \| w(t_k) - w(t_l)\|_0 = 0\}.$$

Hence, $F_n \in \mathscr{F}$ since B is a Borel subset of E and $\| \cdot \|_0$ is a Borel measurable function on E.

Since Brownian motion generated by μ in B exists [3], $\Omega_B \in \mathscr{F}$ (and $P(\Omega_B) = 1$), and $\mathscr{F} \cap \Omega_B$ is the canonical sigma-algebra for B-valued functions we have the existence of Brownian motion in B implying that Brownian motion in E exists as indicated. Further, if $\tilde{M}$ is the closure of H in B under $\| \cdot \|_0$ we have, with probability one, that our motion is in $\tilde{M}$. Since $\tilde{M}$ is a Borel subspace of M (a) holds as M is obviously a closed separable subspace of E_N because H maps continuously in into E_N under the identity.

That C_M is a Frechet space follows since M is a Frechet space. Let $t_j = j/2^N$ ($j = 0, 1,\ldots, 2^N$) and let $\{x_n\}$ be a countable dense subset of M. Let S_N denote the subspace of C_M consisting of functions which are linear on each of the subintervals $[t_{j-1}, t_j]$ with values at t_j in the set $\{x_n\}$. Then $\bigcup_{N=1}^{\infty} S_N$ is a countable dense subset of C_M, and, hence, C_M is separable. To complete the proof of (b) we argue in a standard manner (see, for example [7, Lemma 1-b]).

By [7, Lemma 1] and the above we know that Brownian motion on E generated by μ induces a regular mean zero Gaussian measure P on C_B and since C_B maps continuously into C_M we have P inducing a regular mean zero Gaussian measure on C_M with $P(C_M) = 1$ and hence, on C_E. Hence, (c) holds.

Now assume $\{e_n\}$ is as in (d) and let $\| \cdot \|$ be a continuous semi-norm on M. Then by [7, Lemma 4] we have (5.1) holding with P-probability one for the norm $\| \cdot \|_0$ defined on $B \subseteq M$. Now $\| \cdot \|$ continuous on M implies there exists a $C > 0$ such that $\| \cdot \| \leqslant C \| \cdot \|_0$ on B and hence we have $\sup_{0 \leqslant t \leqslant 1} \|f(t)\| \leqslant \sup_{0 \leqslant t \leqslant 1} C \|f(t)\|_0$ for all f continuous from $[0, \infty)$ into B. Since $P(\Omega_B) = 1$ (5.1) holds with P-probability one for $\| \cdot \|$. That the partial sums in (5.1) converge to $w(t)$ in C_E with P-probability one follows since the partial sums and $w(t)$ are in C_M with probability one and the topology on C_M induced by C_E is a Frechet topology, i.e., C_M is a Frechet space and, hence, is generated by countably many semi-norms of the type found in (5.1).

Now (e) follows immediately from Lemma 4 of [7] since the Hilbert space in C_E which generates P on C_E is the same as the Hilbert space in C_B which generates P on C_B. That is, C_B is a Borel subset of C_E and Borel subsets of C_B are mapped to Borel subsets of C_E under the identity by [1, p. 123] with $P(C_B) = 1$. Hence, $\mathscr{H}$ is the same for C_B as for C_E as it is a unique point set in $C_B \subseteq C_E$ by applying Theorem 3-d to the tight Borel mean zero Gaussian measure P on C_E.

Finally, (f) holds since $\mathscr{K}$ is a compact subset of C_B [7, Lemma 3] and hence in C_E since C_B injects continuously into C_E under the identity map.

6. Strassen's Log Log law for Brownian Motion in E

In [7] we showed how to extend Strassen's law of the iterated logarithm [10] to Brownian motion in a real separable Banach space B generated by a tight mean zero Gaussian measure μ. Here we prove this result when our Brownian motion is in a strict inductive limit of Frechet spaces.

Let d be a distance on the metric space E. A net $\{p_\alpha\}$ in E converges to a set D in E if for each $\epsilon > 0$ there exists an α_0 such that $\alpha \geqslant \alpha_0$ implies $p_\alpha \in \{y: d(y, x) < \epsilon$ for some $x \in D\}$.

Theorem 5. *Let E be a strict inductive limit of Frechet spaces $\{E_n\}$ and let $\{W(t): 0 \leqslant t < \infty\}$ be Brownian motion in E generated by a tight mean zero Gaussian measure μ on E. Suppose μ is generated by H on E, $\mathscr{K}$ is the Hilbert space in C_E which generates P on C_E, and $\mathscr{K}$ is the unit ball of $\mathscr{H}$. For each $t \in [0, 1]$, $s \geqslant 3$, let*

$$\xi_s(t) = \frac{W(st)}{(2s \log \log s)^{1/2}}.$$

Let $\|\cdot\|$ be any continuous semi-norm on E. Then the net $\{\xi_s(t): s \geqslant 3\}$ with P-probability one converges to the compact set $\mathscr{K}$ in the semi-norm $\|f\|_\infty = \sup_{0\leqslant t\leqslant 1} \|f(t)\|$ for $f \in C_E$ and clusters at every point of $\mathscr{K}$ in the semi-norm $\|\cdot\|_\infty$.

Corollary 2. *Assume the setup in Theorem 5 and that E is a Frechet space. Then the net $\{\xi_s(t): s \geqslant 3\}$ with P-probability one converges to the compact set $\mathscr{K}$, and it clusters at every point of $\mathscr{K}$ with P-probability one in the Frechet space C_E.*

Proof of Theorem 5. Let B, $\|\cdot\|_0$, C_B, and Ω_B be as in the proof of Theorem 4. Then with P-probability one our sample paths are in Ω_B and the net $\{\xi_s(t): s \geqslant 3\}$ is in C_B. Since $\|\cdot\|$ is continuous on E it is continuous on $B \subsetneq E$ and hence there is a constant $M > 0$ such that $\| x \| \leqslant M \| x \|_0$ for $x \in B$. Since the net $\{\xi_s(t): s \geqslant 3\}$ converges to $\mathscr{K}$ and clusters at every point of $\mathscr{K}$ with probability one with respect to the norm $\| f \|_{C_B} = \sup_{0 \leqslant t \leqslant 1} \| f(t) \|_0$ on C_B [7, Theorem 1] we have our theorem proved. That is, since $\| f \|_\infty \leqslant M \| f \|_{C_B}$ for all $f \in C_B$ clustering at $\mathscr{K}$ and converging to $\mathscr{K}$ in $\| \cdot \|_{C_B}$ implies the same for $\| \cdot \|_\infty$.

The proof of Corollary 2 is now immediate since convergence in C_E is equivalent to convergence in countably many continuous seminorms on C_E of the type $\| \cdot \|_\infty$ of Theorem 5.

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